Density approach in modelling successive defaults

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Density approach in modelling multi-defaults*

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October 24, 2013

Abstract

We apply the default density framework developed in El Karoui et al. [8] to modelling of multiple defaults, which can be adapted to both top-down and bottom-up models. We present general pricing results and establish links with the classical intensity approach. Explicit models are also proposed by using the methods of change of probability measure or dynamic copula.

1 Introduction

In the credit risk analysis, the dependence of default times is one of the most important issues, for the portfolio credit derivatives, and also for the contagious credit risks management. In the literature, the modelling of multiple credit names is diversified in two directions by using the so-called "bottom-up" and "top-down" models. In the former approach, one first models the marginal distribution of each default time and then the correlation between them, often using the copula models. So the individual default information is taken into account in these models. However, the copula correlation is for a fixed maturity and satisfies no Markovian property. Furthermore, computations can become complicated when it concerns a large number of credit names. The latter one is adapted to the modelling of portfolio credit products of large size and consists of describing directly the cumulative loss process and its dynamics. This approach provides efficiently tractable models for computations. However marginal distributions of default times are relatively neglected in the top-down models.

*This work is partially supported by la chaire Marchés en mutation.

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In this paper, we propose a new method to study credit dependence. Our aim is firstly to propose a dynamic framework for the portfolio credit derivatives, and secondly to make clear the impact of one default event on the other ones. The methodology is based on the default density approach introduced in El Karoui et al. [8] which is suitable for the after-default analysis. We are interested in successive default events, where the before and after default studies adapt naturally. Moreover, this viewpoint allows us to include individual default time information and to obtain pricing formulas for credit portfolio derivatives by using a recursive procedure.

In this context of multiple default times, the market information becomes complicated since it concerns filtrations associated to different default times, besides the "default-free" reference filtration $\mathbb{F}$. The pricing problem consists of computing conditional expectations with respect to this market filtration $\mathbb{G}$, which is the enlarged filtration of $\mathbb{F}$ by all the random times, given different default scenarios. It should be noted that for a portfolio of size n, there exist 2^n possible default events. However, if we consider the ordered set of default times, we can limit ourselves to $n+1$ default scenarios, which reduces largely the computation burden. The main idea is to apply, in a recursive manner, the "before-default" and "after-default" results developed in [8] to the ordered default times and to establish the relationship between stochastic processes in the successive filtrations. This is done under the key hypothesis that the joint conditional survival probability of ordered default times admits a density given the reference filtration $\mathbb{F}$. We study the relationship between the intensity processes of default times and their joint density process. We are also interested in the density under an equivalent change of probability measure. From these results, we analyze the impact of a default on the succeeding ones such as the conditional survival probability given the past defaults, and the contagious jump of the intensity at the default times, etc.

The density framework can also be applied to the non-ordered defaults. By using statistics order, we have a direct link between the density process of a family of non-ordered random times and its increasing ordered permutation. The construction of the non-ordered density process is also closely related to the dynamic copula modelling.

The following of this paper is organized as follows. We present the density framework in Section 2. Section 3 deals with ordered defaults and we explain the relationship with the top-down approach. In Section 4, we discuss the density under a change of probability measure. Section 5 is devoted to non-ordered defaults and contains some dynamic copula examples. We conclude in the last section.

2 Default density framework

In this section, we present the density framework in the credit risk modelling. Let us fix a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ and consider a family of random times $\boldsymbol{\tau} = (\tau_1, \cdots, \tau_n)$ on $(\Omega, \mathcal{A}, \mathbb{P})$ taking values on $\mathbb{R}_+^n$ and representing the default times of n firms on the financial market. In practice, such as for top-down models, we are often interested in the ordered set of $\boldsymbol{\tau}$ for pricing or risk management purposes. We denote the increasing-ordered permutation of $\boldsymbol{\tau}$ by $\boldsymbol{\sigma} = (\sigma_1, \cdots, \sigma_n)$, such that

$$\sigma_1 \leq \sigma_2 \cdots \leq \sigma_n.$$

The idea is to work on the n successive sets $\{t < \sigma_1\}, \{\sigma_1 \leq t < \sigma_2\}, \cdots, \{\sigma_n \leq t\}$ and to update the conditional laws at the arrival of a default: one starts with a filtration $\mathbb{F}$ and, at the first default time σ_1, we enlarge it with that new knowledge to obtain $\mathbb{G}^{(1)}$. Then, we enlarge $\mathbb{G}^{(1)}$ with the knowledge of the second default σ_2 to obtain $\mathbb{G}^{(2)}$, and so on. This progressively arriving information flow will have an impact on the pricing and the risk management problems of the portfolio derivatives, which is the main focus of this paper. The same methodology works with non-ordered defaults. However, in the latter case, the number of default scenarios increases to 2^n, making the study less tractable.

2.1 Reminder on single default

We first recall some results in the case of a single default in [8]. Let τ be a positive random time on the filtered probability space $(\Omega, \mathcal{A}, \mathbb{F}, \mathbb{P})$. We assume that there exists a family of $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+)$-measurable functions $(\omega, u) \to \alpha_t(u)$ such that for any bounded Borel function $f : \mathbb{R}_+ \to \mathbb{R}$,

$$\mathbb{E}[f(\tau)|\mathcal{F}_t] = \int_{\mathbb{R}_+} f(u)\alpha_t(u)du, \quad \forall t \geq 0, \quad \mathbb{P} - a.s. \tag{1}$$

The $\mathbb{F}$-conditional survival probability is given by $S_t(\theta) := \mathbb{P}(\tau > \theta|\mathcal{F}_t) = \int_\theta^\infty \alpha_t(u)du$ for all $t, \theta \geq 0$ and we call the $\mathbb{F}$-survival process the super-martingale $S_t := S_t(t) = \int_t^\infty \alpha_t(u)du$. Roughly speaking, $\alpha_t(u)du = \mathbb{P}(\tau \in du|\mathcal{F}_t)$. So $\alpha(u)$ is the conditional density of τ given the filtration $\mathbb{F}$. The main idea of our approach is that by using the conditional density α, the study in the larger filtration can be done in two steps: before the default, i.e., on the set $\{t < \tau\}$ and after the default, on the set $\{t \geq \tau\}$. The filtration $\mathbb{F}$ is called the reference filtration. In the setting with multiple defaults, we shall update this filtration with the successive knowledge of past defaults.

Let $\mathbb{D} = (\mathcal{D}_t)_{t\geq 0}$ be the smallest right-continuous and complete filtration which makes τ a stopping time. The global market information $\mathbb{G} = (\mathcal{G}_t)_{t\geq 0}$ is the smallest right continuous and complete filtration which contains $\mathcal{F}_t \vee \mathcal{D}_t$. In what follows, we shall write, with an abuse of notation, $\mathcal{F}_t \vee \mathcal{D}_t$ for the regularized filtration.

The pricing problems are related to the computation of $\mathbb{G}$ conditional expectations. Consider a positive and $\mathcal{F}_T \otimes \mathcal{B}(\mathbb{R}_+)$-measurable random variable $Y_T(\cdot)$, T denoting the maturity, then, for any $t \leq T$,

$$\mathbb{E}[Y_T(\tau)|\mathcal{G}_t] = 1_{\{t<\tau\}} \frac{\mathbb{E}\big[\int_t^\infty Y_T(u)\alpha_T(u)du|\mathcal{F}_t\big]}{\mathbb{P}(\tau > t|\mathcal{F}_t)} + 1_{\{\tau \leq t\}} \frac{\mathbb{E}\big[Y_T(\theta)\alpha_T(\theta)\big|\mathcal{F}_t\big]}{\alpha_t(\theta)}\Big|_{\theta=\tau}, \quad a.s. \tag{2}$$

There are two parts in the above formula: the before-default one on the set $\{t < \tau\}$ and the after-default one on the set $\{t \geq \tau\}$. The default timing τ has an impact on the after-default formula, described by the quotient $\mathbb{E}\big[Y_T(\theta)\frac{\alpha_T(\theta)}{\alpha_t(\theta)}\big|\mathcal{F}_t\big]$ evaluated at $\theta = \tau$. The after-default formula in (2) can also be written as

$$1_{\{\tau \leq t\}}\mathbb{E}[Y_T(\tau)|\mathcal{G}_t] = 1_{\{\tau \leq t\}}\mathbb{E}_{\mathbb{P}^\theta}[Y_T(\theta)|\mathcal{F}_t]\big|_{\theta=\tau}$$

where

$$\frac{d\mathbb{P}^\theta}{d\mathbb{P}}\Big|_{\mathcal{F}_T} = \frac{\alpha_T(\theta)}{\alpha_0(\theta)}$$

So the impact of default can be interpreted as a change of probability measure.

An explicit relationship exists between the density of τ and the widely-used default intensity. Recall that the $\mathbb{G}$-intensity of τ is the positive $\mathbb{G}$-adapted process $\lambda^{\mathbb{G}}$ such that $(1_{\{\tau \leq t\}} - \int_0^t \lambda_s^{\mathbb{G}} ds, t \geq 0)$ is a $\mathbb{G}$-martingale. The $\mathbb{F}$-intensity of τ is the positive $\mathbb{F}$-adapted process $\lambda^{\mathbb{F}}$ such that $(1_{\{\tau \leq t\}} - \int_0^{t\wedge\tau} \lambda_s^{\mathbb{F}} ds, t \geq 0)$ is a $\mathbb{G}$-martingale and satisfies $\lambda_t^{\mathbb{F}} 1_{\{\tau > t\}} = \lambda_t^{\mathbb{G}}$. The intensity $\lambda_t^{\mathbb{F}}$ can be completely deduced from the density by

$$\lambda_t^{\mathbb{F}} = \frac{\alpha_t(t)}{S_t}, \quad t \geq 0. \tag{3}$$

However, from the intensity, we can obtain part of the density

$$\alpha_t(\theta) = \mathbb{E}[\lambda_\theta^{\mathbb{G}}|\mathcal{F}_t] = \mathbb{E}[\lambda_\theta^{\mathbb{F}} 1_{\{\theta<\tau\}}|\mathcal{F}_t], \quad t \leq \theta \tag{4}$$

In classical intensity models, one often assumes that the immersion property holds between $\mathbb{F}$ and $\mathbb{G}$, i.e., that $\mathbb{F}$-martingales remain $\mathbb{G}$-martingales. This condition, also called the H-hypothesis, is equivalent to $\mathbb{P}(\tau > t|\mathcal{F}_t) = \mathbb{P}(\tau > t|\mathcal{F}_\infty)$, so that $S_t = \exp(-\int_0^t \lambda_s^{\mathbb{F}} ds)$, and the "after-default" density satisfies

$$\alpha_t(\theta) = \alpha_\theta(\theta), \quad t > \theta \tag{5}$$

Hence, from the intensity process, we obtain the whole density family for all positive t and θ under immersion. Another important consequence of (5) is that the after-default conditional expectation becomes

$$E[Y_T(\tau)|\mathcal{G}_t]1_{\{\tau \leq t\}} = 1_{\{\tau \leq t\}}\mathbb{E}\big[Y_T(\theta)\big|\mathcal{F}_t\big]_{\theta=\tau}$$

2.2 Conditional density for ordered defaults

We consider now a multidefault setting with ordered defaults $\boldsymbol{\sigma} = (\sigma_1, \cdots, \sigma_n)$ and extend the density framework to this case. As we have explained previously, a family of increasingly enlarged filtrations is associated to the successive default times. We firstly state the density hypothesis with respect to the reference filtration $\mathbb{F}$, and then consider the filtrations containing default information.

2.2.1 Density w.r.t. default-free filtration

Let $\mathbb{F} = (\mathcal{F}_t)_{t\geq 0}$ be a reference filtration on $(\Omega, \mathcal{A}, \mathbb{P})$ satisfying the usual conditions and representing the "default free" information. We present firstly the fundamental hypothesis on the existence of the $\mathbb{F}$-density of the ordered defaults family $\boldsymbol{\sigma}$.

Hypothesis 2.1 The conditional distribution of $\boldsymbol{\sigma} = (\sigma_1, \cdots, \sigma_n)$ given $\mathbb{F}$ admits a density with respect to the Lebesgue measure[1], i.e., there exists a family of $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+^n)$-measurable functions $(\omega, \boldsymbol{u}) \to \alpha_t(\boldsymbol{u}, \omega)$ where $\boldsymbol{u} = (u_1, \cdots, u_n) \in \mathbb{R}_+^n$, such that for any (bounded) Borel function $f : \mathbb{R}_+^n \to \mathbb{R}$,

$$\mathbb{E}[f(\boldsymbol{\sigma})|\mathcal{F}_t] = \int_{\mathbb{R}_+^n} f(\boldsymbol{u})\alpha_t(\boldsymbol{u})d\boldsymbol{u}, \quad \forall t \geq 0, \, \mathbb{P} - a.s. \tag{6}$$

where $d\boldsymbol{u}$ denotes $du_1 \cdots du_n$. We call the family $\alpha(\boldsymbol{u})$ the $\mathbb{F}$-*conditional density* (or the *density* if no ambiguity) of $\boldsymbol{\sigma}$.

In the following of this paper, we always assume this hypothesis. The density hypothesis implies that there are no simultaneous defaults, that is, $\sigma_i \neq \sigma_j$ a.s. for all $i \neq j$. This is a standard hypothesis for the multiple default times. We note that $\alpha(\boldsymbol{u})$ is null outside the set $\{u_1 \leq \cdots \leq u_n\}$. For any fixed $\boldsymbol{u} \in \mathbb{R}_+^n$, the process $(\alpha_t(\boldsymbol{u}), t \geq 0)$ is an $\mathbb{F}$-martingale. The joint conditional survival law is given, for any $\boldsymbol{\theta} = (\theta_1, \cdots, \theta_n) \in \mathbb{R}_+^n$, by

$$S_t(\boldsymbol{\theta}) := \mathbb{P}(\boldsymbol{\sigma} > \boldsymbol{\theta}|\mathcal{F}_t) = \int_{\theta_1}^{\infty} du_1 \cdots \int_{\theta_n}^{\infty} du_n \, \alpha_t(\boldsymbol{u}) = \int_{\boldsymbol{\theta}}^{\infty} \alpha_t(\boldsymbol{u})d\boldsymbol{u}$$

where the notation $\boldsymbol{\sigma} > \boldsymbol{\theta}$ stands for $\sigma_i > \theta_i$ for all $i \in \{1, \cdots, n\}$.

In the following, for any $\boldsymbol{u}$, we use the notation $\boldsymbol{u}_{(k:p)}$ to denote the vector $(u_k, \cdots, u_p)$, $d\boldsymbol{u}_{(k:p)}$ to denote $du_k \dots du_p$ and

$$\int_t^{\infty} f(\boldsymbol{u}_{(k:n)})d\boldsymbol{u}_{(k:n)} := \int_t^{\infty} du_k \cdots \int_t^{\infty} du_n \, f(u_k, \dots, u_n)$$

[1]This can be generalized to any non-negative non-atomic measure which is invariant by permutation.

We also use $\boldsymbol{u}_{(p)}$ for $\boldsymbol{u}_{(1:p)}$. Furthermore, $\boldsymbol{u}_{(i:i-1)}$ and $\boldsymbol{u}_{(0)}$ are null vectors.

For the subfamily $\boldsymbol{\sigma}_{(k)} := (\sigma_1, \ldots, \sigma_k)$ where $k \le n$, the marginal density $\alpha_t^{(k)}(\cdot)$ of $\boldsymbol{\sigma}_{(k)}$ with respect to $\mathcal{F}_t$ is the $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+^k)$-measurable function obtained from the joint density of $\boldsymbol{\sigma}$ as a partial integral by

$$\alpha_t^{(k)}(\boldsymbol{u}_{(k)}) = \int_{\mathbb{R}_+^{n-k}} \alpha_t(\boldsymbol{u}) d\boldsymbol{u}_{(k+1:n)}. \tag{7}$$

We denote $\alpha_{t,t}^{(k)}(\boldsymbol{u}_{(k)}) := \int_t^\infty \alpha_t(\boldsymbol{u}) d\boldsymbol{u}_{(k+1:n)}$, so that

$$\mathbb{P}(\sigma_{k+1} > t | \mathcal{F}_t) = \int_{\mathbb{R}_+^k} \alpha_{t,t}^{(k)}(\boldsymbol{u}_{(k)}) d\boldsymbol{u}_{(k)}.$$

The equality $\alpha_t^{(k)}(\boldsymbol{u}_{(k-1)}, t) = \alpha_{t,t}^{(k)}(\boldsymbol{u}_{(k-1)}, t)$ holds for all positive $\boldsymbol{u}_{(k-1)}$ and t.

2.2.2 Density w.r.t. global filtration

For a single default τ, the global information is the progressive enlargement of the filtration $\mathbb{F}$ by the default time τ. In the multidefault case, the information contains the successive enlargements of filtrations by the ordered defaults. Let us begin by making precise the default filtrations.

The default information arrives progressively with successive defaults. For any $i \in \{1, \cdots, n\}$, we denote by $\mathbb{D}^i = (\mathcal{D}_t^i)_{t \ge 0}$ the filtration associated with σ_i, which corresponds to the information concerning the i^{th} default. So the increasing filtrations $\mathbb{D}^{(i)} = (\mathcal{D}_t^{(i)})_{t \ge 0} := \mathbb{D}^1 \vee \cdots \vee \mathbb{D}^i$ represent the cumulative information flow of the first i defaults, notably, whether the first i defaults have occurred and the timings of the past default events. In other words, at each default, we update the information by adding the σ-algebra generated by σ_i.

The global information $\mathbb{G}^{(n)} := \mathbb{F} \vee \mathbb{D}^{(n)}$ includes both default and default-free information. For any i, we define the intermediate filtration $\mathbb{G}^{(i)} = (\mathcal{G}_t^{(i)})_{t \ge 0} := \mathbb{F} \vee \mathbb{D}^{(i)}$ and we set for convenience $\mathbb{G}^{(0)} = \mathbb{F}$. Notice that $\mathbb{G}^{(i)}$ coincides with $\mathbb{G}^{(n)}$ stopped at the corresponding default, i.e., $\mathcal{G}_t^{(i)} = \mathcal{G}_{t \wedge \sigma_i}^{(n)}$, $t \ge 0$. Any $\mathcal{G}_t^{(n)}$-measurable random variable X can be written in the form

$$X_t = \sum_{i=0}^{n} 1_{\{\sigma_i \le t < \sigma_{i+1}\}} X_t^i(\boldsymbol{\sigma}_{(i)})$$

where $X_t^i(\cdot)$ is $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+^i)$-measurable, and $\sigma_0 = 0$, $\sigma_{n+1} = \infty$ by convention.

We introduce the $\mathcal{G}_t^{(n)}$-conditional law of $\boldsymbol{\sigma}$ defined by:

$$\mathbb{E}[f(\boldsymbol{\sigma}) | \mathcal{G}_t^{(n)}] = \int_{\mathbb{R}_+^n} f(\boldsymbol{u}) \mu_t^{(n)}(d\boldsymbol{u}) \tag{8}$$

where

$$\mu_t^{(n)}(d\boldsymbol{u}) = \sum_{i=0}^{n} 1_{\{\sigma_i \le t < \sigma_{i+1}\}} \frac{\alpha_t(\boldsymbol{u})d\boldsymbol{u}_{(i+1:n)}}{\alpha_{t,t}^{(i)}(\boldsymbol{u}_{(i)})} \delta_{\boldsymbol{\sigma}_{(i)}}(d\boldsymbol{u}_{(i)}) \tag{9}$$

with δ denoting the Dirac measure. Observe that the conditional law $\mu^{(n)}$ admits a regime change at each default time, on the set $\{\sigma_i \le t < \sigma_{i+1}\}$, it depends on the $\mathbb{F}$-conditional law of the first i defaults $\boldsymbol{\sigma}_{(i)}$ and on their timings.

In particular, in the single default case (when $n = 1$) recalled in Section 2.1,

$$\mu_t^{(1)}(u)du = 1_{\{t<\tau\}} \frac{\alpha_t(u)du}{\int_t^\infty \alpha_t(u)du} + 1_{\{t\ge\tau\}}\delta_\tau(du)$$

and the conditional expectation (2) can be written as

$$\mathbb{E}[Y_T(\tau)|\mathcal{G}_t] = \int_{\mathbb{R}_+} \mathbb{E}\big[Y_T(u)\frac{\alpha_T(u)}{\alpha_t(u)}|\mathcal{F}_t\big]\mu_t^{(1)}(du).$$

We now generalize this result in the multidefault setting, which will be useful in the sequel.

Proposition 2.2 *Let $Y_T(\cdot)$ be a positive $\mathcal{F}_T \otimes \mathcal{B}(\mathbb{R}_+^n)$-measurable function on $\Omega \times \mathbb{R}_+^n$, then for any $t \le T$,*

$$\mathbb{E}[Y_T(\boldsymbol{\sigma})|\mathcal{G}_t^{(n)}] = \int_{\mathbb{R}_+^n} \mathbb{E}^{\mathbb{P}^{\boldsymbol{u}}}[Y_T(\boldsymbol{u})|\mathcal{F}_t]\mu_t^{(n)}(d\boldsymbol{u}) \tag{10}$$

where $\mathbb{P}^{\boldsymbol{u}}$ is defined by $\frac{d\mathbb{P}^{\boldsymbol{u}}}{d\mathbb{P}}\big|_{\mathcal{F}_T} = \frac{\alpha_T(\boldsymbol{u})}{\alpha_0(\boldsymbol{u})}$, or equivalently,

$$\mathbb{E}[Y_T(\boldsymbol{\sigma})|\mathcal{G}_t^{(n)}] = \sum_{i=0}^{n} 1_{\{\sigma_i \le t < \sigma_{i+1}\}} \frac{\int_t^\infty d\boldsymbol{u}_{(i+1:n)}\, \mathbb{E}[Y_T(\boldsymbol{u})\alpha_T(\boldsymbol{u})|\mathcal{F}_t]}{\alpha_{t,t}^{(i)}(\boldsymbol{u}_{(i)})}\Big|_{\boldsymbol{u}_{(i)}=\boldsymbol{\sigma}_{(i)}}, \ a.s. \tag{11}$$

Proof: We proceed in a recursive way. Indeed, the formula (2) adapts naturally to the successive defaults; in particular, applying the before-defaut part of formula (2), on the set $\{\sigma_i \le t < \sigma_{i+1}\}$, to the subfamily of remaining defaults $\boldsymbol{\sigma}_{(i+1:n)} = (\sigma_{i+1}, \cdots, \sigma_n)$ and the corresponding filtration $\mathcal{G}_t^{(i)}$ (as $\mathcal{F}_t$ in (2)) leads to

$$\begin{aligned}
&1_{\{\sigma_i \le t < \sigma_{i+1}\}}\mathbb{E}[Y_T(\boldsymbol{\sigma})|\mathcal{G}_t^{(n)}] \\
&= 1_{\{\sigma_i \le t < \sigma_{i+1}\}} \frac{\mathbb{E}[\int_t^\infty Y_T(\boldsymbol{u})\alpha_T^{(i+1:n)|i}(\boldsymbol{u}_{(i+1:n)})d\boldsymbol{u}_{(i+1:n)}|\mathcal{G}_t^{(i)}]}{\mathbb{P}(\sigma_{i+1} > t|\mathcal{G}_t^{(i)})}\Big|_{\boldsymbol{u}_{(i)}=\boldsymbol{\sigma}_{(i)}}
\end{aligned}$$

where $\alpha^{(i+1:n)|i}$ denotes the $\mathbb{G}^{(i)}$ density of $\boldsymbol{\sigma}_{(i+1:n)}$ and is given, on the set $\{\sigma_j \le t < \sigma_{j+1}\}$ for any $j = 0, \cdots, i-1$, by

$$\alpha_t^{(i+1:n)|i}(\boldsymbol{u}_{(i+1:n)}) = \frac{\int_t^\infty d\boldsymbol{s}_{(j+1:i)}\alpha_t(\boldsymbol{\sigma}_{(j)}, \boldsymbol{s}_{(j+1:i)}, \boldsymbol{u}_{(i+1:n)})}{\alpha_{t,t}^{(j)}(\boldsymbol{\sigma}_{(j)})} \tag{12}$$

and on the set $\{\sigma_i \le t < \sigma_{i+1}\}$,

$$\alpha_t^{(i+1:n)|i}(\boldsymbol{u}_{(i+1:n)}) = \frac{\alpha_t(\boldsymbol{\sigma}_{(i)}, \boldsymbol{u}_{(i+1:n)})}{\alpha_t^{(i)}(\boldsymbol{\sigma}_{(i)})}. \tag{13}$$

We then use the after-default part of (2) to the default subfamily $\boldsymbol{\sigma}_{(i)}$ and the reference filtration $\mathcal{F}_t$ to obtain, on the set $\{\sigma_i \le t < \sigma_{i+1}\}$,

$$\begin{aligned}
&\mathbb{E}[\int_t^\infty Y_T(\boldsymbol{u})\alpha_T^{(i+1:n)|i}(\boldsymbol{u}_{(i+1:n)})d\boldsymbol{u}_{(i+1:n)}|\mathcal{G}_t^{(i)}]\big|_{\boldsymbol{u}_{(i)}=\boldsymbol{\sigma}_{(i)}} \\
&= \frac{\mathbb{E}[\alpha_T^{(i)}(\boldsymbol{u}_{(i)})\int_t^\infty Y_T(\boldsymbol{u})\alpha_T^{(i+1:n)|i}(\boldsymbol{u}_{(i+1:n)})d\boldsymbol{u}_{(i+1:n)}|\mathcal{F}_t]}{\alpha_t^{(i)}(\boldsymbol{u}_{(i)})}\Big|_{\boldsymbol{u}_{(i)}=\boldsymbol{\sigma}_{(i)}} \\
&= \frac{\mathbb{E}[\int_t^\infty Y_T(\boldsymbol{u})\alpha_T(\boldsymbol{u})d\boldsymbol{u}_{(i+1:n)}|\mathcal{F}_t]}{\alpha_t^{(i)}(\boldsymbol{u}_{(i)})}\Big|_{\boldsymbol{u}_{(i)}=\boldsymbol{\sigma}_{(i)}}
\end{aligned}$$

Moreover,

$$\mathbb{P}(\sigma_{i+1} > t|\mathcal{G}_t^{(i)}) = 1_{\{\sigma_i > t\}} + 1_{\{\sigma_i \le t\}}\frac{\alpha_{t,t}^{(i)}(\boldsymbol{\sigma}_{(i)})}{\alpha_t^{(i)}(\boldsymbol{\sigma}_{(i)})}$$

This leads to (11). Finally, we obtain the equivalent equality (10) by using the conditional law $\mu_t^{(n)}$ given by (9). □

The contagion effect is contained in our model through two aspects. The first one is that the density α does not factorize, as it would be the case if the default are (conditionally) independent. The second one is that the conditional densities given the global information depend explicitly on the past defaults. There are very few papers where this last effect is taken into account. In literature, the intensity changes after the occurrence of default taking into account the number of defaults, but does not keep in memory the timing of each past default (as it is the case in the papers of Arnsdorff and Halperin [1], Laurent et al. [16], Herbertsson [14], etc.). This allows these models to be Markovian, which is not the case in our paper.

We are interested in the law of the k^{th} default, or the joint law of the the first k defaults in the portfolio. Firstly, the marginal survival law of σ_k is given by (8) as

$$\begin{aligned}
\mathbb{P}(\sigma_k > T|\mathcal{G}_t^{(n)}) &= \int_{\mathbb{R}_+^n} 1_{\{u_k > T\}}\mu_t^{(n)}(d\boldsymbol{u}) \\
&= \int_{\mathbb{R}_+^n} 1_{\{u_k > T\}}\sum_{i=0}^{k-1} 1_{\{\sigma_i \le t < \sigma_{i+1}\}}\frac{\alpha_t(\boldsymbol{u})d\boldsymbol{u}_{(i+1:n)}}{\alpha_{t,t}^{(i)}(\boldsymbol{u}_{(i)})}\,\delta_{\boldsymbol{\sigma}_{(i)}}(d\boldsymbol{u}_{(i)})
\end{aligned} \tag{14}$$

More generally, by the fact $\mathcal{G}_t^{(k)} = \mathcal{G}_{t\wedge\sigma_k}^{(n)}$, the marginal conditional law of $\boldsymbol{\sigma}_{(k)}$ given $\mathcal{G}_t^{(k)}$ and given $\mathcal{G}_t^{(n)}$ coincide on $\{t < \sigma_k\}$. Denoting by μ^k the $\mathcal{G}^{(k)}$ conditional law of $\boldsymbol{\sigma}_{(k)}$, defined as

$\mathbb{E}[f(\boldsymbol{\sigma}_{(k)})|\mathcal{G}_t^{(k)}] = \int_{\mathbb{R}_+^n} f(\boldsymbol{u})\mu_t^{(k)}(d\boldsymbol{u})$, we see that it differs from the partial sum of $\mu^{(n)}$ (summands from $i=0$ to k in (9)) only on the last set $\{\sigma_k \le t\}$, more precisely

$$\mu_t^{(k)}(d\boldsymbol{u}) = \sum_{i=0}^{k-1} 1_{\{\sigma_i \le t < \sigma_{i+1}\}} \frac{\alpha_t(\boldsymbol{u})d\boldsymbol{u}_{(i+1:n)}}{\alpha_{t,t}^{(i)}(\boldsymbol{u}_{(i)})} \delta_{\boldsymbol{\sigma}_{(i)}}(d\boldsymbol{u}_{(i)}) + 1_{\{\sigma_k \le t\}} \frac{\alpha_t(\boldsymbol{u})d\boldsymbol{u}_{(k+1:n)}}{\alpha_{t,t}^{(k)}(\boldsymbol{u}_{(k)})} \delta_{\boldsymbol{\sigma}_{(k)}}(d\boldsymbol{u}_{(k)}) \quad (15)$$

and we have $\mathbb{P}(\sigma_k > T|\mathcal{G}_t^{(n)}) = \int_{\mathbb{R}_+^n} 1_{\{u_k > T\}}\mu_t^{(k)}(d\boldsymbol{u})$.

3 Successive defaults and top-down models

In this section, we investigate the links between successive defaults and the top-down models. Recall that $\boldsymbol{\tau}$ are a family of random times and $\boldsymbol{\sigma}$ are the associated ordered sequence.

3.1 Default information — ordered defaults and cumulative loss

The information of the ordered default times can also be given by the knowledge of the default counting process

$$N_t := \sum_{i=1}^{n} 1_{\{\tau_i \le t\}}$$

which also equals $N_t = \sum_{i=1}^n 1_{\{\sigma_i \le t\}}$. At a given time $t \ge 0$, the information generated by the random variable N_t, i.e. $\sigma(N_t)$ gives the number of defaults up to time t; and the filtration $\mathbb{D}^N = (\mathcal{D}_t^N)_{t\ge 0}$ generated by the counting process N, i.e. $\mathcal{D}_t^N = \sigma(N_s, s \le t)$ gives the number of past defaults together with their timings. In addition, $\mathbb{D}^N$ coincides with the information flow generated by the ordered default times, i.e., $\mathcal{D}_t^{(n)} = \mathcal{D}_t^N$. Clearly, the global information $\mathbb{G}^N = (\mathcal{G}_t^N)_{t\ge 0}$ including $\mathbb{F}$ and $\mathbb{D}^N$ coincides with $\mathbb{G}^{(n)}$ and satisfies $\mathcal{G}_{t\wedge\sigma_i}^N = \mathcal{G}_t^{(i)}$. In literature, the counting process N is often supposed to be Markovian. This assumption appears to be convenient in some cases. Nevertheless, our objective in this paper is to analyze the impact of past defaults in more detail, so we take into account the timing of default events.

A useful observation gives the link between the ordered defaults and the counting process N: for any $k = 1, \cdots, n$ and any $t \ge 0$,

$$\{N_t < k\} = \{\sigma_k > t\}.$$

By the marginal law of ordered defaults and the fact that $\mathbb{G}^N$ and $\mathbb{G}^{(n)}$ coincide, we obtain the conditional loss distribution for $t \le T$,

$$P_k(t,T) := \mathbb{P}(N_T \le k|\mathcal{G}_t^N) = \mathbb{P}(\sigma_{k+1} > T|\mathcal{G}_t^N) = \int_{\mathbb{R}_+^n} 1_{\{u_{k+1} > T\}}\mu_t^{(k+1)}(d\boldsymbol{u}) \quad (16)$$

where $\mu^{(k)}$ is the $\mathbb{G}^{(k)}$ conditional law of $\boldsymbol{\sigma}_{(k)}$ given by (15). Obviously $\mathbb{P}(N_T = k|\mathcal{G}_t^N) = P_k(t,T) - P_{k-1}(t,T)$. Note that the loss distribution (16) depends not only on the number of defaults, but also on the occurrence timing of each default.

In particular, letting $k=1$ leads to the following familiar result for the first default:

$$\mathbb{P}(\sigma_1 > T|\mathcal{G}_t^N) = 1_{\{\sigma_1>t\}}\frac{\mathbb{P}(\sigma_1 > T|\mathcal{F}_t)}{\mathbb{P}(\sigma_1 > t|\mathcal{F}_t)} = 1_{\{\sigma_1>t\}}\frac{\int_T^\infty d\boldsymbol{u}\,\alpha_t(\boldsymbol{u})}{\int_t^\infty d\boldsymbol{u}\,\alpha_t(\boldsymbol{u})}.$$

For a CDO tranche pricing, the standard hypothesis on the market is the identical and constant recovery rate R (equal to 40% in practice). So $L_T = (1-R)N_T$. Then using the equality $(N_T-K)^+ = \int_K^\infty 1_{\{N_T>u\}}du$ and (16) allow to obtain $\mathbb{E}[(N_T-K)^+|\mathcal{G}_t^N]$, which is the key term for the CDO pricing.

3.2 Links with intensity approach

Most top-down models in the literature follow the intensity approach. In this section, we establish the link between density and intensity: the $\mathbb{F}$-density of $\boldsymbol{\sigma}$ will give the full knowledge of the intensity of the loss process, the reverse is not always true unless under the H-hypothesis.

The loss intensity is the $\mathbb{G}^N$-adapted process λ^N such that $(N_t - \int_0^t \lambda_s^N ds, t \geq 0)$ is a $\mathbb{G}^N$-martingale. This quantity is often used in the top-down models to characterize the loss distribution. In the following, we show that the loss intensity equals the sum of all the ordered default intensities.

Recall that the $\mathbb{G}^{(i)}$-intensity of σ_i is the positive $\mathbb{G}^{(i)}$-adapted process λ^i such that $(M_t^i := 1_{\{\sigma_i\leq t\}} - \int_0^t \lambda_s^i ds, t\geq 0)$ is a $\mathbb{G}^{(i)}$-martingale. It is well known that M^i is stopped at σ_i and the intensity satisfies $\lambda_t^i = 0$ on $\{t\geq\sigma_i\}$. The following lemma shows that the $\mathbb{G}^{(i)}$-intensity of σ_i coincides with its $\mathbb{G}^N$-intensity.

Lemma 3.1 *For $i = 1,\cdots,n$, any $\mathbb{G}^{(i)}$-martingale stopped at σ_i is a $\mathbb{G}^N$-martingale.*

Proof: We prove that any $\mathbb{G}^{(1)}$-martingale stopped at σ_1 is a $\mathbb{G}^{(2)}$-martingale. The result will follow. Let M be a $\mathbb{G}^{(1)}$-martingale stopped at σ_1, i.e. $M_t = M_{t\wedge\sigma_1}$ for any $t\geq 0$. For $s<t$,

$$\mathbb{E}[M_{t\wedge\sigma_1}|\mathcal{G}_s^{(2)}] = 1_{\{\sigma_2\leq s\}}M_{\sigma_1} + 1_{\{s<\sigma_2\}}\frac{\mathbb{E}[M_{t\wedge\sigma_1}1_{\{s<\sigma_2\}}|\mathcal{G}_s^{(1)}]}{\mathbb{P}(s<\sigma_2|\mathcal{G}_s^{(1)})}$$

It remains to note that

$$\mathbb{E}[M_{t\wedge\sigma_1}1_{\{s<\sigma_2\}}|\mathcal{G}_s^{(1)}] = 1_{\{s<\sigma_1\}}\mathbb{E}[M_{t\wedge\sigma_1}|\mathcal{G}_s^{(1)}] + 1_{\{\sigma_1\leq s\}}\mathbb{E}[M_{\sigma_1}1_{\{s<\sigma_2\}}|\mathcal{G}_s^{(1)}]\,.$$

The martingale property of M yields to

$$1_{\{s<\sigma_1\}}\mathbb{E}[M_{t\wedge\sigma_1}|\mathcal{G}_s^{(1)}] = 1_{\{s<\sigma_1\}}M_{s\wedge\sigma_1}$$

It is obvious that

$$1_{\{\sigma_1\le s\}}\mathbb{E}[M_{\sigma_1}1_{\{s<\sigma_2\}}|\mathcal{G}_s^{(1)}] = 1_{\{\sigma_1\le s\}}M_{\sigma_1}\mathbb{P}(s<\sigma_2|\mathcal{G}_s^{(1)}).$$

Since $\sigma_2 > s$ on $\{\sigma_1 > s\}$, we obtain $1_{\{s<\sigma_1\}}\mathbb{P}(s<\sigma_2|\mathcal{G}_s^{(1)}) = 1_{\{s<\sigma_1\}}$. The result follows. □

Proposition 3.2 *The loss intensity equals the sum of $\mathbb{G}^{(i)}$-intensities of all ordered defaults, i.e., for any $t\ge 0$,*

$$\lambda_t^N = \sum_{i=1}^n \lambda_t^i, \quad a.s. \tag{17}$$

where

$$\lambda_t^i = 1_{\{\sigma_{i-1}\le t<\sigma_i\}}\frac{\alpha_{t,t}^{(i)}(\boldsymbol{u}_{(i)})}{\alpha_{t,t}^{(i-1)}(\boldsymbol{u}_{(i-1)})}\Big|_{\substack{\boldsymbol{u}_{(i-1)}=\boldsymbol{\sigma}_{(i-1)}\\ u_i=t}}, \ a.s. \tag{18}$$

Remark 3.3 The intensity of the i^{th} default depends on the "after default" part of the density, or more precisely, on the density $\alpha_t(\boldsymbol{u})$ after $(i-1)^{\text{th}}$ default with $u_{i-1} < t \le u_i$, and also on the first $i-1$ defaults $\boldsymbol{\sigma}_{(i-1)}$.

Proof: Since $(1_{\{\sigma_i\le t\}} - \int_0^t \lambda_s^i ds, t\ge 0)$ is a $\mathbb{G}^{(i)}$-martingale stopped at σ_i, it is a $\mathbb{G}^N$-martingale. The sum $(N_t - \int_0^t \sum_{i=1}^n \lambda_s^i ds, t\ge 0)$ is a $\mathbb{G}^N$-martingale. So $\lambda^N = \sum_{i=1}^n \lambda^i$. For (18), we use a recursive method with $\mathbb{G}^{(i)} = \mathbb{D}^i \vee \mathbb{G}^{(i-1)}$. As a consequence of (3),

$$\lambda_t^i = 1_{\{t<\sigma_i\}}\frac{\alpha_t^{i|i-1}(t)}{S_t^{i|i-1}},$$

where $S_t^{i|i-1} = \mathbb{P}(\sigma_i > t|\mathcal{G}_t^{(i-1)}) = \int_t^\infty \alpha_t^{i|i-1}(u)du$. Using (12) and (13), we obtain that the $\mathbb{G}^{(i-1)}$-density of σ_i satisfies $1_{\{t<\sigma_{i-1}\}}\alpha_t^{i|i-1}(t) = 0$ and on the set $\{\sigma_{i-1}\le t\}$,

$$\alpha_t^{i|i-1}(t) = \frac{\alpha_{t,t}^{(i)}(\boldsymbol{u}_{(i)})}{\alpha_t^{(i-1)}(\boldsymbol{u}_{(i-1)})}\Big|_{\substack{\boldsymbol{u}_{(i-1)}=\boldsymbol{\sigma}_{(i-1)}\\ u_i=t}}$$

In addition

$$S_t^{i|i-1} = 1_{\{t<\sigma_{i-1}\}} + 1_{\{\sigma_{i-1}\le t\}}\frac{\alpha_{t,t}^{(i-1)}(\boldsymbol{u}_{(i-1)})}{\alpha_t^{(i-1)}(\boldsymbol{u}_{(i-1)})}\Big|_{\boldsymbol{u}_{(i-1)}=\boldsymbol{\sigma}_{(i-1)}}$$

so the result follows. □

We note that the $\mathbb{G}^{(i)}$-intensity of σ_i is given in the form

$$\lambda_t^i = 1_{\{\sigma_{i-1}\le t<\sigma_i\}}\lambda_t^{i,\mathbb{F}}(\boldsymbol{\sigma}_{(i-1)}) \tag{19}$$

where $\lambda_t^{i,\mathbb{F}}(\cdot)$ is $\mathcal{F}_t \otimes \mathcal{B}(\mathbb{R}_+^{i-1})$-measurable. Without loss of generality, we can suppose that $\lambda_t^{i,\mathbb{F}}(\boldsymbol{u}_{(i-1)}) = 0$ if $\boldsymbol{u}_{(i-1)}$ is outside the set $\{u_1 \leq \cdots \leq u_{i-1}\}$ or if $t < u_{i-1}$ or $t > u_i$.

Some explicit models of loss intensity have been proposed in literature where λ^N is supposed to be a function of N (e.g. Brigo et al. [3], Cont and Minca [4], Filipović et al. [10] and Sidenius et al. [17]). For example, λ^N depends on an auxiliary Markov chain in Frey and Backhaus [12], and on some contagion factors in Arnsdorf and Halperin [1], (but not on the default timings in these models). In Errais et al. [7], the loss intensity depends on the timing of defaults using Hawkes processes.

3.3 Successive defaults and immersion

For the successive defaults, the immersion holds between all the successive filtrations, that is, between $\mathbb{G}^{(i)}$ and $\mathbb{G}^{(i+1)}$ for any $i = 0, \cdots, n-1$ if and only if $\mathbb{F}$ is immersed in $\mathbb{G}^N$ (see Ehlers and Schönbucher [6]). We characterize the immersion property in the density framework for ordered defaults.

Proposition 3.4 *The immersion property between $\mathbb{F}$ and $\mathbb{G}^N$ is equivalent to the following conditions: for any $i = 1, \cdots, n$ and any $\boldsymbol{u} \in \mathbb{R}_+^n$ such that $u_1 \leq \cdots \leq u_n$, it holds*

$$\alpha_t^{(i)}(\boldsymbol{u}_{(i)}) = \alpha_{u_i}^{(i)}(\boldsymbol{u}_{(i)}), \quad \forall t > u_i \tag{20}$$

where $\alpha^{(i)}$ is the $\mathbb{F}$-density of $\boldsymbol{\sigma}_{(i)}$ given by (7).

Proof: The result holds for $n = 1$. We shall prove for $n = 2$ and the general result follows by recurrence. In fact, it's easy to verify that under the conditions (20), we have $\alpha_t^{1|0}(u) = \alpha_u^{1|0}(u)$ and $\alpha_t^{2|1}(u) = \alpha_u^{2|1}(u)$ for $t \geq u$, which are equivalent to the immersion between $\mathbb{F}$ and $\mathbb{G}^{(1)}$ and between $\mathbb{G}^{(1)}$ and $\mathbb{G}^{(2)}$ respectively and hence to the immersion between $\mathbb{F}$ and $\mathbb{G}^{(2)}$. $\square$

Using the intensity approach, we can construct a family of ordered random times satisfying the immersion property. Let λ^i be a family of positive $\mathbb{G}^i$-adapted intensity processes and assume that $\int_0^\infty \lambda_s^i ds = +\infty$, then an immediate recurrence establishes that $\mathbb{F}$ is immersed in $\mathbb{G}^N$ if

$$\sigma_i = \inf\{t \geq 0 \, : \, \int_0^t \lambda_s^i ds \geq \eta_i\}$$

where η_i is independent of $\mathbb{F} \vee \mathbb{G}^{(i-1)}$, hence of $\eta_1, \ldots, \eta_{i-1}$. Note that, since the intensity of the i^{th} default σ_i is null before σ_{i-1}, using the notation in (19), we get

$$\sigma_i = \inf\{t \geq \sigma_{i-1} \, : \, \int_{\sigma_{i-1}}^t \lambda_s^{i,\mathbb{F}}(\boldsymbol{\sigma}_{(i-1)}) ds \geq \eta_i\}.$$

The case where $(\eta_1, \cdots, \eta_n)$ is a family of mutually independent uni-exponential random variables corresponds to the successive Cox model described in [6].

The loss distribution $P_k(t,T)$ whose general form is given in (16) has a more familiar form under immersion. The following result generalizes a well-known result in the single default case. Note that under immersion, the probability of having less than k defaults in the portfolio depends only on $\lambda^{k+1,\mathbb{F}}(\boldsymbol{\sigma}_k)$, that is, the intensity of σ_{k+1} or equivalently, the loss intensity restricted to the set $\{\sigma_k \leq t < \sigma_{k+1}\}$.

Proposition 3.5 *We assume the H-hypothesis between $\mathbb{F}$ and $\mathbb{G}^N$. Then the loss distribution is given by*

$$P_k(t,T) = 1_{\{t<\sigma_{k+1}\}}\mathbb{E}\Big[\exp\big\{-\int_t^T \lambda_s^{k+1,\mathbb{F}}\big(\boldsymbol{\sigma}_{(k)}\big)ds\big\}|\mathcal{G}_t^{(k)}\Big]. \tag{21}$$

Proof: By the H-hypothesis and a recursive argument,

$$P_k(t,T) = \mathbb{E}[1_{\{\sigma_{k+1}>T\}}|\mathcal{G}_t^N] = \mathbb{E}[1_{\{\sigma_{k+1}>T\}}|\mathcal{G}_t^{(k+1)}] = 1_{\{\sigma_{k+1}>t\}}\frac{\mathbb{E}[S_T^{k+1|k}|\mathcal{G}_t^{(k)}]}{S_t^{k+1|k}}$$

where $S_t^{k+1|k} = \mathbb{P}(\sigma_{k+1} > t|\mathcal{G}_t^{(k)})$. Since the immersion property holds between $\mathbb{G}^{(k)}$ and $\mathbb{G}^{(k+1)}$ by [6], $S_t^{k+1|k} = \exp(-\int_0^t \lambda_s^{k+1}(\boldsymbol{\sigma}_{(k)})ds)$, the result follows. □

The following proposition gives the density of $\boldsymbol{\sigma}$ in terms of the marginal intensities. We shall revisit it by using the change of probability viewpoint in the next section.

Proposition 3.6 *If the immersion property between $\mathbb{F}$ and $\mathbb{G}^N$, then*

$$\alpha_t(\boldsymbol{u}) = \begin{cases} \mathbb{E}[\alpha_{u_n}(\boldsymbol{u})|\mathcal{F}_t],\ 0 \leq t \leq u_n \\ \alpha_{u_n}(\boldsymbol{u}),\ t > u_n \end{cases}$$

where

$$\alpha_{u_n}(\boldsymbol{u}) = \prod_{i=1}^n \lambda_{u_i}^{i,\mathbb{F}}\big(\boldsymbol{u}_{(i-1)}\big)\exp\big\{-\int_{u_{i-1}}^{u_i}\lambda_s^{i,\mathbb{F}}\big(\boldsymbol{u}_{(i-1)}\big)ds\big\}. \tag{22}$$

Proof: The case where $n=1$ holds. We shall prove the proposition for $n=2$ and the general result follows. Considering σ_2 by recurrence leads

$$\alpha_t^{2|1}(u_2) = \mathbb{E}[\lambda_{u_2}^{2,\mathbb{F}}(\sigma_1)\exp\big(-\int_0^{u_2}\lambda_s^{2,\mathbb{F}}(\sigma_1)ds\big)|\mathcal{G}_t^{(1)}]$$

Identifying both sides of the equality on the set $\{t<\sigma_1\}$ by (12) and (2) implies

$$\begin{aligned}\int_t^\infty du_1\alpha_t(u_1,u_2) &= \mathbb{E}[1_{\{\sigma_1>t\}}\lambda_{u_2}^{2,\mathbb{F}}(\sigma_1)\exp(-\int_0^{u_2}\lambda_s^{2,\mathbb{F}}(\sigma_1)ds)|\mathcal{F}_t] \\ &= \mathbb{E}[\int_t^\infty du_1\alpha_{u_2}^{1|0}(u_1)\lambda_{u_2}^{2,\mathbb{F}}(u_1)\exp(-\int_0^{u_2}\lambda_u^{2,\mathbb{F}}(u_1)du)|\mathcal{F}_t]\end{aligned}$$

Under immersion between $\mathbb{F}$ and $\mathbb{G}^{(1)}$, $\alpha^{1|0}_{u_2}(u_1) = \alpha^{1|0}_{u_1}(u_1) = \lambda^{1,\mathbb{F}}_{u_1}\exp(-\int_0^{u_1}\lambda^{1,\mathbb{F}}_s ds)$, which concludes the proof. $\square$

4 Change of probability measure

We have explored in [8] and [9] the close relationship between the density family and the methodology of change of probability. On the one hand, the density is affected by a change of probability; on the other hand, we can construct density processes using a change of probability measure.

4.1 Density under a change of probability

We begin from a probability space $(\Omega, \mathcal{A}, \mathbb{P})$ where the immersion holds between $\mathbb{F}$ and $\mathbb{G}^{(n)}$ and $\boldsymbol{\sigma}$ has the $\mathbb{F}$-density $\alpha(\cdot)$. We are interested in the density process of $\boldsymbol{\sigma}$ under an equivalent probability measure where the immersion property holds no longer.

Let W be an $\mathbb{F}$-Brownian motion and hence a $\mathbb{G}^{(n)}$-Brownian motion. Let Q be a positive $\mathbb{G}^{(n)}$-martingale with expectation 1, the solution of the SDE

$$dQ_t = Q_{t-}(\Psi_t dW_t + \sum_{i=1}^n \Phi^i_t dM^i_t), \quad Z_0 = 1$$

where $(M^i_t = 1_{\{\sigma_i \le t\}} - \int_0^{t\wedge\sigma_i}\lambda^{i,\mathbb{F}}_s(\boldsymbol{\sigma}_{(i-1)})ds, t \ge 0)$ are $\mathbb{G}^{(n)}$-martingales of pure jump, Ψ and Φ^i are $\mathbb{G}^{(n)}$-predictable processes which can be written in the form $\Psi_t = \sum_{i=0}^n 1_{\{\sigma_i < t \le \sigma_{i+1}\}}\psi^i_t(\boldsymbol{\sigma}_{(i)})$ and $\Phi^i_t = \sum_{k=0}^{i-1} 1_{\{\sigma_k < t \le \sigma_{k+1}\}}\phi^{i,k}_t(\boldsymbol{\sigma}_{(k)})$ with $\phi^{i,k} > -1$. Then

$$Q_t = \sum_{i=0}^n 1_{\{\sigma_i \le t < \sigma_{i+1}\}} q^i_t(\boldsymbol{\sigma}_{(i)})$$

where for $i = 1, \cdots, n-1$ and $t > u_i$,

$$q^i_t(\boldsymbol{u}_{(i)}) = q^i_{u_i}(\boldsymbol{u}_{(i)}) \cdot \exp\Big(\int_{u_i}^t \psi^i_s(\boldsymbol{u}_{(i)})dW_s - \frac{1}{2}\int_{u_i}^t \psi^i_s(\boldsymbol{u}_{(i)})^2 ds - \int_{u_i}^t \phi^{i+1,i}_s(\boldsymbol{u}_{(i)})\lambda^{i+1,\mathbb{F}}_s(\boldsymbol{u}_{(i)})ds\Big)$$

and for $t > u_n$,

$$q^n_t(\boldsymbol{u}) = q^n_{u_n}(\boldsymbol{u})\exp\Big(\int_{u_n}^t \psi^n_s(\boldsymbol{u})dW_s - \frac{1}{2}\int_{u_n}^t \psi^n_s(\boldsymbol{u})^2 ds\Big),$$

with initial values

$$q^0_0 = 1, \quad \text{and} \quad q^i_{u_i}(\boldsymbol{u}_{(i)}) = q^{i-1}_{u_i}(\boldsymbol{u}_{(i-1)})\big(1 + \phi^{i,i-1}_{u_i}(\boldsymbol{u}_{(i-1)})\big),\ i = 1, \cdots n.$$

Let $\mathbb{Q}$ be the probability measure defined by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = Q_t \quad \text{on} \quad \mathcal{G}_t^{(n)}.$$

Then the density of $\boldsymbol{\sigma}$ under $\mathbb{Q}$ is given by

$$\alpha_t^{\mathbb{Q}}(\boldsymbol{u}) = \begin{cases} \frac{1}{Q_t^{\mathbb{F}}}\mathbb{E}\Big[q_{u_n}^{n-1}\big(\boldsymbol{u}_{(n-1)}\big)\big(1+\phi_{u_n}^{n,n-1}\big(\boldsymbol{u}_{(n-1)}\big)\big)\alpha_{u_n}\big(\boldsymbol{u}_{(n)}\big)|\mathcal{F}_t\Big], & t \le u_n \\ \frac{1}{Q_t^{\mathbb{F}}}\, q_t^n(\boldsymbol{u})\alpha_t(\boldsymbol{u}), \quad t > u_n \end{cases} \tag{23}$$

where $Q^{\mathbb{F}}$ is the restriction of Q on $\mathbb{F}$, i.e. $Q_t^{\mathbb{F}} := \mathbb{E}[Q_t|\mathcal{F}_t]$.

4.2 Dynamic copula

We can also start from the unconditional law of $\boldsymbol{\sigma}$ and construct a conditional density in a dynamic way. This gives a dynamic copula viewpoint: we shall diffuse the initial dependence structure of defaults. The following results are extensions to [9, Section 5] and appeals to the change of probability measure.

Let us begin from the elementary case where $\boldsymbol{\sigma}$ is independent of the filtration $\mathbb{F}$ and admits a probability density $\alpha_0 : \mathbb{R}_+^n \to \mathbb{R}_+$, that is, $\mathbb{P}(\boldsymbol{\sigma} > \boldsymbol{\theta}|\mathcal{F}_t) = \mathbb{P}(\boldsymbol{\sigma} > \boldsymbol{\theta}) = \int_{\boldsymbol{\theta}}^{\infty} \alpha_0(\boldsymbol{u})d\boldsymbol{u}$. We consider the following change of probability. Let $(\beta_t(\boldsymbol{u}), t \ge 0)$ be a family of $\mathbb{F}$-martingales with $\beta_0(\boldsymbol{u}) = 1$ for all $\boldsymbol{u} \in \mathbb{R}_+^n$. Then $(\beta_t(\boldsymbol{\sigma}), t \ge 0)$ is a martingale w.r.t. the filtration $\mathbb{G}^{\boldsymbol{\sigma}} = \mathbb{F} \vee \sigma(\boldsymbol{\sigma})$ and defines a new probability measure $\mathbb{Q}$ by

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{G}_t^{\boldsymbol{\sigma}}} = \beta_t(\boldsymbol{\sigma}).$$

The density of $\boldsymbol{\sigma}$ under $\mathbb{Q}$ is given by

$$\alpha_t^{\mathbb{Q}}(\boldsymbol{u}) = \frac{1}{m_t^{\beta}}\beta_t(\boldsymbol{u})\alpha_0(\boldsymbol{u})$$

where $m_t^{\beta} = \int_0^{\infty} \beta_t(\boldsymbol{u})\alpha_0(\boldsymbol{u})d\boldsymbol{u}$. In particular, we have $\mathbb{Q}(\boldsymbol{\sigma} > \boldsymbol{\theta}) = \mathbb{P}(\boldsymbol{\sigma} > \boldsymbol{\theta})$, the unconditional law of $\boldsymbol{\sigma}$ remains unchanged under the two probability measures.

At the time $t = 0$, the density function α_0 represents the initial correlation between the default times $\boldsymbol{\sigma}$, which can be modelled for example by using a copula function. The dynamic dependence is introduced through the change of probability measure under the new probability $\mathbb{Q}$, using the normalized $\mathbb{Q}$-martingale $\frac{\beta_t(\boldsymbol{u})}{m_t^{\beta}}$. We can view $\beta_t(\boldsymbol{u})\alpha_0(\boldsymbol{u})$ as a non-normalized density under $\mathbb{Q}$, obtained by a linear transformation from the initial density under $\mathbb{P}$. Then the normalization factor m_t^{β} introduces a nonlinear dependence of $\alpha_t^{\mathbb{Q}}(\boldsymbol{u})$ with respect to the initial density. We need here a family of $\mathbb{F}$-martingales instead of a $\mathbb{G}^{(n)}$-martingale as in Section 4.1.

A change of probability as above allows us to construct a sequence of successive defaults $\boldsymbol{\sigma}$ with given density $\alpha_t(\boldsymbol{u})$ or intensities $\boldsymbol{\lambda} = (\lambda^1, \cdots, \lambda^n)$. More precisely, we start from a family of random times $\boldsymbol{\sigma}$ on $(\Omega, \mathcal{A}, \mathbb{G}^{\boldsymbol{\tau}}, \mathbb{P})$ which is independent of $\mathbb{F}$, so its $\mathbb{F}$-density $\alpha(\boldsymbol{u})$ coincides with its initial value $\alpha_0(\boldsymbol{u})$. Define the new probability measure $\mathbb{Q}$ on $(\Omega, \mathbb{G}^{\boldsymbol{\tau}})$ by

$$\frac{d\mathbb{Q}}{d\mathbb{P}}\Big|_{\mathcal{G}_t^{\boldsymbol{\sigma}}} = \frac{\alpha_t(\boldsymbol{\sigma})}{\alpha_0(\boldsymbol{\sigma})},$$

then $\boldsymbol{\sigma}$ admits under $\mathbb{Q}$ the density $\alpha_t(\boldsymbol{u})$. If we are given instead a family of intensity processes of the form $\lambda_t^i = \lambda_t^{i,\mathbb{F}}(\boldsymbol{\sigma}_{(i-1)})$, then it is possible to construct a family of ordered default times $\boldsymbol{\sigma} = (\sigma_1, \cdots, \sigma_n)$ with intensity λ_i for σ_i. We define

$$\alpha_t(\boldsymbol{u}) = \mathbb{E}\Big[\prod_{i=1}^n \lambda_{u_i}^{i,\mathbb{F}}\big(\boldsymbol{u}_{(i-1)}\big) \exp\big\{ - \int_{u_{i-1}}^{u_i} \lambda_s^{i,\mathbb{F}}\big(\boldsymbol{u}_{(i-1)}\big) ds\big\} | \mathcal{F}_t\Big] \quad a.s. \tag{24}$$

and $\frac{d\mathbb{Q}}{d\mathbb{P}}\big|_{\mathcal{G}_t^{\boldsymbol{\sigma}}} = \frac{\alpha_t(\boldsymbol{\sigma})}{\alpha_0(\boldsymbol{\sigma})}$, then $\boldsymbol{\sigma}$ admits $\boldsymbol{\lambda}$ as intensities and $\alpha(\boldsymbol{u})$ as density under $\mathbb{Q}$. In addition, the immersion property holds between $\mathbb{F}$ and $\mathbb{G}^{(n)}$ under $\mathbb{Q}$. In the case where $\boldsymbol{\lambda}$ do not depend on the past defaults, the construction can be done as in [13]. In our case, the situation is more complex since the change of probability measure will be affected by the timing of the defaults.

5 Bottom-up models and density framework

In the bottom-up models, we are interested in the individual credit names where the density framework can also be adapted.

5.1 Non-ordered default times

Let us consider the family of non-ordered default times $\boldsymbol{\tau} = (\tau_1, \ldots, \tau_n)$ and assume the density hypothesis of $\boldsymbol{\tau}$ w.r.t. $\mathbb{F}$. That is, there exists a family of positive $\mathbb{F}$-martingales $\beta(\boldsymbol{\theta})$ such that for any bounded Borel function $f : \mathbb{R}_+^n \to \mathbb{R}_+$,

$$\mathbb{E}[f(\boldsymbol{\tau})|\mathcal{F}_t] = \int_{\mathbb{R}_+^n} f(\boldsymbol{\theta})\beta_t(\boldsymbol{\theta})d\boldsymbol{\theta}, \quad t \geq 0.$$

Consider the ordered defaults $\boldsymbol{\sigma}$ to be the increasing permutation of $\boldsymbol{\tau}$. Then there exists an explicit relationship between $\beta(\cdot)$ and the density $\alpha(\cdot)$ of $\boldsymbol{\sigma}$ by using the statistics order. For any $\boldsymbol{u} \in \mathbb{R}_+^n$ such that $u_1 \leq \cdots \leq u_n$ and any $t \geq 0$,

$$\alpha_t(u_1, \cdots, u_n) = 1_{\{u_1 \leq \cdots \leq u_n\}} \sum_{\Pi} \beta_t\big(u_{\Pi(1)}, \cdots, u_{\Pi(n)}\big) \tag{25}$$

where $(\Pi(1), \cdots, \Pi(n))$ is a permutation of $(1, \cdots, n)$. The equality (25) allows us to incorporate the individual default information into the density of the ordered default vector. In particular, if $\boldsymbol{\tau}$ is exchangeable, that is (see e.g. Frey and McNeil [11]), if $(\tau_1, \cdots, \tau_n) \stackrel{\mathrm{d}}{=} (\tau_{\Pi(1)}, \cdots, \tau_{\Pi(n)})$ for any permutation where $\stackrel{\mathrm{d}}{=}$ signifies the equality in distribution, then

$$\alpha_t(u_1, \cdots, u_n) = 1_{\{u_1 \leq \cdots \leq u_n\}} \, n! \, \beta_t(u_1, \cdots, u_n).$$

This implies that all subfamilies of $\boldsymbol{\tau}$ with the same cardinal has the same distribution. In other words, the portfolio is homogeneous.

The density approach can also be applied directly to the non-ordered defaults in a similar way as for the ordered ones. However, it is necessary to consider 2^n possible default scenarios and the recursive before-default and after-default methodology no longer adapts.

5.2 Density models and dynamic copula

In this subsection, we give several explicit models for the conditional probability $G_t(\boldsymbol{\theta}) := \mathbb{P}(\boldsymbol{\tau} > \boldsymbol{\theta} | \mathcal{F}_t)$ of non-ordered defaults $\boldsymbol{\tau}$, with which we can deduce the density.

The following example is a backward one based on the Cox-process model. The correlation structure is fixed for the final time by a copula function and $G_t(\boldsymbol{\theta})$ is obtained by taking conditional expectation.

Example 5.1 Let τ_i, $i \in \{1, \ldots, n\}$ be defined as in the Cox process model in Lando [15]. That is, $\tau_i = \inf\{t : \Phi_t^i \geq \xi_i\}$ where Φ^i is an $\mathbb{F}$-adapted increasing process satisfying $\Phi_0^i = 0$ and $\lim_{t \to \infty} \Phi_t^i = +\infty$, ξ_i is a $\mathcal{A}$-measurable random variable of exponential law with parameter 1 and is independent of $\mathcal{F}_\infty$. In this model, the marginal survival process is given by $G_t^i = \mathbb{P}(\tau_i > t | \mathcal{F}_\infty) = e^{-\Phi_t^i}$. So the H-hypothesis holds between $\mathbb{F}$ and $\mathbb{F} \vee \mathbb{H}^i$. Let the correlation of defaults be represented by a copula function $\mathcal{C} : \mathbb{R}_+^n \to \mathbb{R}_+$ such that $\mathbb{P}(\boldsymbol{\tau} > \boldsymbol{\theta} | \mathcal{F}_\infty) = \mathcal{C}(G_{\theta_1}^1, \ldots, G_{\theta_n}^n)$. Then

$$G_t(\boldsymbol{\theta}) = \mathbb{E}\big[\mathcal{C}(G_{\theta_1}^1, \ldots, G_{\theta_n}^n) | \mathcal{F}_t\big].$$

By choosing different copula functions, we obtain a large family of joint densities.

The Gaussian copula model is the standard market model for the CDO pricing, where the correlation between defaults is described by a standard gaussian random variable representing the common market factor. We now generalize the Gaussian copula model in a dynamic way. In the following, the first example is a backward one where the factor is a CIR process and the second one is a forward constructive model (see also Crépey et al. [5]).

Example 5.2 Consider a family of processes $X = (X^1, \cdots, X^n)$ where $X^i = Y^0 + Y^i$ ($i = 1, \cdots, n$) is a fundamental process of the i^{th} firm depending on two factors: the process Y^0 can be interpreted as a common factor of the market and Y^i is the individual factor of each firm which are independent Cox-Ingersoll-Ross processes

$$dY_t^i = \kappa_i(\mu_i - Y_t^i)dt + \sigma_i\sqrt{Y_t^i}dB_t^i, \;\; Y_0^i > 0, \quad i = 0, 1 \cdots, n.$$

We assume moreover that $2\kappa_i\mu_i > \sigma_i^2$ so that Y^i does not vanish. Let the filtration $\mathbb{F}$ be generated by the multi-dimensional Brownian motion $B = (B^0, B^1, \cdots, B^n)$. Let $T \geq 0$ be a terminal time. We define the conditional survival probability for $0 \leq t < T$ and $\boldsymbol{\theta} = (\theta_1, \cdots, \theta_n) \in \mathbb{R}_+^n$ by

$$G_t(\boldsymbol{\theta}) = \mathbb{P}(\boldsymbol{\tau} > \boldsymbol{\theta}|\mathcal{F}_t) = \mathbb{E}[e^{-\boldsymbol{\theta}\cdot X_T}|\mathcal{F}_t] = \mathbb{E}[e^{-\sum_{i=0}^n \theta_i Y_T^i}|\mathcal{F}_t]$$

where $u_0 = u_1 + \cdots + u_n$. Then classical results on affine processes yield

$$G_t(\boldsymbol{\theta}) = \exp\big(-\sum_{i=0}^n \varphi_i(T-t, \theta_i)Y_t^i - \sum_{i=0}^n \psi_i(T-t, \theta_i)\big).$$

with φ_i and ψ_i being functions $\mathbb{R}_+ \times \mathbb{R}_+ \to \mathbb{R}$ given explicitly by

$$\varphi_i(s, v) = \frac{2\kappa_i v}{(2\kappa_i + \sigma_i^2 v)e^{\kappa_i s} - \sigma_i^2 v}, \quad \psi_i(s, v) = \frac{2\kappa_i\mu_i}{\sigma_i^2}\big(\ln\frac{(2\kappa_i + \sigma_i^2 v)e^{\kappa_i s} - \sigma_i^2 v}{2\kappa_i} - \kappa_i s\big).$$

The correlation between default times is characterized by the process Y^0. The case $Y^0 = 0$ provides an example where the default times are independent. Moreover, given the process Y^0, the default times satisfy the standard conditional independence condition.

In particular, for an homogeneous portfolio, $\kappa_i = \kappa$, $\mu_i = \mu$ and $\sigma_i = \sigma$ for $i = 0, \cdots, n$. So the functions satisfy $\varphi_i = \varphi$ and $\psi_i = \psi$ and

$$G_t(\boldsymbol{\theta}) = \exp\big(-\sum_{i=0}^n \varphi(T-t, \theta_i)Y_t^i - \sum_{i=0}^n \psi(T-t, \theta_i)\big).$$

Example 5.3 Let h_i be a family of increasing functions mapping $\mathbb{R}_+$ into $\mathbb{R}$, $B = (B^i, i = 1, \ldots, n)$ an n-dimensional standard Brownian motion and Y a random variable independent of $\mathbb{F}$. We set

$$\tau_i = (h_i)^{-1}(\sqrt{1-\rho_i^2}\int_0^\infty f_i(s)dB_s^i + \rho_i Y)$$

for $\rho_i \in (-1, 1)$ and f_i a family of deterministic square-integrable functions. An immediate extension of the Gaussian model leads to

$$\mathbb{P}(\boldsymbol{\tau} > \boldsymbol{\theta}|\mathcal{F}_t \vee \sigma(Y)) = \prod_{i=1}^n \Phi\left(\frac{1}{\sigma_i(t)}\left(m_t^i - \frac{h_i(\theta_i) - \rho_i Y}{\sqrt{1-\rho_i^2}}\right)\right)$$

where $m_t^i = \int_0^t f_i(s) dB_s^i$ and $\sigma_i^2(t) = \int_t^\infty f_i^2(s) ds$. It follows that

$$\mathbb{P}(\boldsymbol{\tau} > \boldsymbol{\theta} | \mathcal{F}_t) = \int_{-\infty}^{\infty} \prod_{i=1}^{n} \Phi\left(\frac{1}{\sigma_i(t)} \left(m_t^i - \frac{h_i(\theta_i) - \rho_i y}{\sqrt{1-\rho_i^2}} \right) \right) f_Y(y) dy$$

where f_Y denotes the probability density function of Y. Note that, in this setting, the random times $\boldsymbol{\tau}$ are conditionally independent given the factor Y, similar as in the standard Gaussian copula model. In the particular case $t = 0$, choosing f_i so that $\sigma_i(0) = 1$, and Y with a standard Gaussian law, we obtain

$$P(\boldsymbol{\tau}_i > \boldsymbol{\theta}) = \int_{-\infty}^{\infty} \prod_{i=1}^{n} \Phi\left(-\frac{h_i(\theta_i) - \rho_i y}{\sqrt{1-\rho_i^2}} \right) \varphi(y) dy$$

which corresponds, by construction, to the standard Gaussian copula ($h_i(\tau_i) = \sqrt{1-\rho_i^2} X_i + \rho_i Y$, where X_i, Y are independent standard Gaussian variables).

Relaxing the independence condition on the components of the process B leads to more sophisticated examples.

6 Conclusion

In this paper, we have applied the density approach to multiple default events. Under the hypothesis on the existence of joint density process with respect to the reference filtration $\mathbb{F}$, we have deduced dynamics of pricing processes for credit portfolio products.

The study is based on the before-default and after-default analysis and allows us to examine in detail the impact of one default event on the remaining credit names such as the contagious jump of the default intensity. Furthermore, the pricing formulas are given on different default scenarios and hence make clear the instantaneous change of a financial product due to the default events.

The dependence structures between default times are represented by their $\mathbb{F}$ conditional density process and we have proposed several modelling methods. The idea is to diffuse a static correlation structure at the initial time to achieve a "dynamic correlation". The density approach provides a new vision on the default dependence problems. Under this theoretical framework, some explicit models of joint density process may be studied in more detail for further practical use.

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